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The Real Zeros of the Bernoulli Polynomials

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1. INTRODUCTION

The problem of finding the number and position of the real zeros of the Bernoulli polynomials has been considered by a number of authors over the past 75 years (see, e.g., [4, 5, 6, 8, 13, 14, 17]). Let $B_n(x)$, $n \geqslant 0$, denote the Bernoulli polynomial of degree n and let $B_n := B_n(0)$ be the nth Bernoulli number (see, e.g., [1]). These polynomials can be defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi.$$

In order to discuss the work already done on this problem, and our contributions to it, we state here some well-known properties of the Bernoulli polynomials (see, e.g., [15])

$$B_n(x) = \sum_{s=0}^{n} \binom{n}{s} 2^{-s} \left(x - \frac{1}{2}\right)^{n-s} D_s, \qquad n \geqslant 0, \tag{1.1}$$

where

$$D_s = 2(1 - 2^{s-1}) B_s, \qquad s \geqslant 0. \tag{1.2}$$

Therefore, each $B_n(x)$ is monic and has exact degree n,

$$B_n(1+x) - B_n(x) = nx^{n-1}, \qquad n \geqslant 0 \tag{1.3}$$

$$B_n(1-x) = (-1)^n B_n(x), \qquad n \geqslant 0 \tag{1.4}$$

$$B_n'(x) = nB_{n-1}(x), \qquad n \geqslant 1. \tag{1.5}$$

Now (1.3) and (1.4) imply that $B_{2n+1}(0) = B_{2n+1}(\frac{1}{2}) = B_{2n+1}(1) = 0$, $n \geqslant 1$. Using (1.5), Rolle's Theorem on $B_{2n+1}(x)$, and (1.4), we see that

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$B_{2n}(x)$ has one real zero in each of the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ which we call r_{2n} and s_{2n} where $s_{2n} := 1 - r_{2n}$ and $0 < r_{2n} < \frac{1}{2}$.

Nörlund [15, p. 22] showed that $0, \frac{1}{2}$, and 1 are the only zeros of $B_{2n+1}(x)$, $n \geqslant 1$, in $[0, 1]$. He also showed [16, p. 131] that for $B_{2n}(x)$ and $n \geqslant 1$, r_{2n} and s_{2n} satisfy $\frac{1}{6} < r_{2n} < \frac{1}{4}$, hence $\frac{3}{4} < s_{2n} < \frac{5}{6}$ (see also [13, p. 534]). J. Lense [14] and A. M. Ostrowski [17] showed that the sequence $\{r_{2n}\}$ is monotonically inreasing to $\frac{1}{4}$. D. H. Lehmer [13] gave the more precise inequality $\frac{1}{4} - 2^{-2n-1}\pi^{-1} < r_{2n} < \frac{1}{4}$. K. Inkeri [8] showed that $0, \frac{1}{2}$, and 1 are the only rational zeros of $B_{2n+1}(x)$, $n \geqslant 1$. He also considered in some detail the number and position of the real zeros of $B_n(x)$ outside the interval $[0, 1]$. Inkeri also gave an asymptotic estimate for the number of real zeros of $B_n(x)$ and, in addition, gave upper and lower bounds for the real zeros of $B_n(x)$ outside the interval $[0, 1]$. These estimates are, as claimed by the author, valid for "large" values of n.

No extensive table of the real or complex zeros of the Bernoulli polynomials has been published to date, although D. H. Lehmer, in 1967, computed the real and complex zeros of $B_n(x)$ up to $n = 48$ using his circle method, and Leon J. Lander, in 1968, computed the zeros of $B_n(x)$, again up to $n = 48$, using a general purpose factorization routine (double precision) on a CDC 6400. These computations were remarkably accurate up to about $n = 42$, although no special effort was made in either case to verify the zeros by high-accuracy methods [2, 11].

In this paper, we confine the discussion to the real zeros of $B_n(x)$. Since each $B_n(x)$ is symmetric about the line $x = \frac{1}{2}$ (cf. (1.4)), we consider only the nonnegative real zeros. The remaining real zeros of $B_n(x)$ can then be obtained using (1.4).

Although the complex zeros are also of some interest, the results are of a different nature and therefore will be the subject of another paper. Some preliminary results in this direction have been obtained jointly with Professor R. S. Varga.

In Section 2 we give an empirical result for calculating the number of real zeros of $B_n(x)$, which is valid for $1 \leqslant n \leqslant 200$. In Section 3, we give some inequalities which provide upper and lower bounds for $|E_{2n}|$ and $|B_{2n}|$, where E_{2n} and B_{2n} are the Euler and Bernoulli numbers, respectively. In Section 4, we give simple expressions for computing $B_n(m+q)$ where $m \geqslant 1$ is an integer and $q = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}$, or 1. These expressions, along with Newton's method and the method of false position, provide, in most cases, simple lower and upper bounds for approximating the real zeros of $B_n(x)$ outside the interval $[0, 1]$. We use the degree n of the Bernoulli polynomial to divide the study into four cases, namely $n \equiv 0$, 1, 2, and 3 (mod 4), and each case is discussed separately in Sections 5 to 8. In Section 5 we give a new result which permits a precise count of the number of real zeros of $B_{4n}(x)$. We also investigate the irregular occurrence of a pair of real zeros of $B_{4n}(x)$ in the interval $(M+\frac{3}{4},\ M+1)$. Here M is the largest positive integer such that $B_{4n}(x)$ has real zeros in the interval $(M, M+1)$. In Section 7 we present the "crossover" phenomenon for the real zeros of $B_{4n+2}(x)$ in the intervals $[m, m+1]$, $m \geqslant 2$. In Section 8, we present a theorem which improves Inkeri's upper and lower bounds for estimating certain zeros of $B_{4n+3}(x)$. In Section 9, we describe the method of computation of the zeros of $B_n(x)$ and in Tables IV and V give a listing of the positive real zeros for $3 \leqslant n \leqslant 117$.

2. The Number of Real Zeros of $B_n(x)$

It is well known [15, p. 19] that all the zeros of $B_n(x)$, $1 \leqslant n \leqslant 5$, are real. Each polynomial $B_n(x)$, $n \geqslant 6$, has complex zeros which occur as "quartets" in the complex plane that are symmetric about the real axis and the line $\operatorname{Re} z = \frac{1}{2}$. That is, if $z = s + it$ is a zero of $B_n(z + \frac{1}{2})$ with $s \geqslant 1$ then so are $z = s - it$ and $z = -s \pm it$.

Inkeri [8, p. 12] has shown that the number R_n of real zeros of $B_n(x)$ has the asymptotic limit $R_n/n \sim (2/\pi e)$ $(n \to \infty)$. Inkeri's proof involves a sign change argument and Stirling's formula and requires a separate study of each of the four cases $n \equiv 0, 1, 2$, and 3 (mod 4). His analysis, however, does not provide a precise count for the number of real zeros of $B_n(x)$. H. Delange [5] gives a method of determining the exact number of real zeros of $B_{4n}(x)$, in most cases.

We give here an empirical method for determining the *exact* number, R_n, of real zeros of $B_n(x)$ up to $n = 200$. The increase in the number of real zeros is not monotonic (see Table I); however, there is a nearly regular pattern to the increase in the number of real zeros of $B_n(x)$ which is described later in this section. This pattern is abruptly broken at $n = 116$ as can be verified from Table I. The pattern is again broken for $n = 179$ which was verified using BERNSCAN (described below). These are the only exceptions for $n \leqslant 200$. Up to $n = 117$, the exact value can be verified directly from Tables IV and V. The method of computation of the real zeros given in Tables IV and V is described in Section 9.

One can count the number of real zeros of $B_n(x)$ for values of n much larger than $n = 117$ simply by using the Lemmas of Section 4 and noting the sign changes of $B_n(x)$ on the intervals $[m, m+1]$, $m \geqslant 1$. We have developed a FORTRAN program called BERNSCAN (described in more detail in Section 5) which computes (single precision) the value of $B_n(x)$ for any specified value of x and for (at least) $n \leqslant 1000$. This can be done at equally spaced points on any interval containing real zeros of $B_n(x)$. Determining the sign changes in $B_n(x)$ in this way will give an exact value for

TABLE I

Table of Values of R_n, $1 \leqslant n \leqslant 200$

n	R_n	k	n	R_n	k	n	R_n	k	n	R_n	k
1	1	0	51	15	9	101	25	19	151	39	28
2	2	0	52	16	9	102	26	19	152	40	28
3	3	0	53	17	9	103	27	19	153	37	29
4	4	0	54	14	10	104	28	19	154	38	29
5	5	0	55	15	10	105	29	19	155	39	29
6	2	1	56	16	10	106	26	20	156	40	29
7	3	1	57	17	10	107	27	20	157	41	29
8	4	1	58	18	10	108	28	20	158	38	30
9	5	1	59	15	11	109	29	20	159	39	30
10	6	1	60	16	11	110	30	20	160	40	30
11	7	1	61	17	11	111	27	21	161	41	30
12	4	2	62	18	11	112	28	21	162	42	30
13	5	2	63	19	11	113	29	21	163	43	30
14	6	2	64	16	12	114	30	21	164	40	31
15	7	2	65	17	12	115	31	21	165	41	31
16	8	2	66	18	12	116	28	22	166	42	31
17	5	3	67	19	12	117	29	22	167	43	31
18	6	3	68	20	12	118	30	22	168	44	31
19	7	3	69	17	13	119	31	22	169	41	32
20	8	3	70	18	13	120	32	22	170	42	32
21	9	3	71	19	13	121	33	22	171	43	32
22	6	4	72	20	13	122	30	23	172	44	32
23	7	4	73	21	13	123	31	23	173	45	32
24	8	4	74	22	13	124	32	23	174	42	33
25	9	4	75	19	14	125	33	23	175	43	33
26	10	4	76	20	14	126	34	23	176	44	33
27	7	5	77	21	14	127	31	24	177	45	33
28	8	5	78	22	14	128	32	24	178	46	33
29	9	5	79	23	14	129	33	24	179	43	34
30	10	5	80	20	15	130	34	24	180	44	34
31	11	5	81	21	15	131	35	24	181	45	34
32	12	5	82	22	15	132	32	25	182	46	34
33	9	6	83	23	15	133	33	25	183	47	34
34	10	6	84	24	15	134	34	25	184	44	35
35	11	6	85	21	16	135	35	25	185	45	35
36	12	6	86	22	16	136	36	25	186	46	35
37	13	6	87	23	16	137	33	26	187	47	35
38	10	7	88	24	16	138	34	26	188	48	35
39	11	7	89	25	16	139	35	26	189	49	35
40	12	7	90	22	17	140	36	26	190	46	36
41	13	7	91	23	17	141	37	26	191	47	36
42	14	7	92	24	17	142	38	26	192	48	36
43	11	8	93	25	17	143	35	27	193	49	36
44	12	8	94	26	17	144	36	27	194	50	36
45	13	8	95	27	17	145	37	27	195	47	37
46	14	8	96	24	18	146	38	27	196	48	37
47	15	8	97	25	18	147	39	27	197	49	37
48	12	9	98	26	18	148	36	28	198	50	37
49	13	9	99	27	18	149	37	28	199	51	37
50	14	9	100	28	18	150	38	28	200	48	38

R_n, up to $n=1000$. This process has been completed, and reported herein, up to $n=200$.

We sought a pattern to determine the values of n for which the value of k increases by one. In other words, for which values of n does $B_n(x)$ obtain an additional "quartet" of complex zeros? For $n \leqslant 200$ these values are $n=6$, 12, 17, 22, 27, 33, 38, 43, 48, 54, 59, 64, 69, 75, 80, 85, 90, 96, 101, 106, 111, 116, 122, 127, 132, 137, 143, 148, 153, 158, 164, 169, 174, 179, 184, 190, 195, and 200. Taking differences of successive values in the above sequence yields the pattern

6, 5, 5, 5, 6, 5, 5, 5, 6, 5, 5, 5, 6, 5, 5, 5, 6, 5, 5, 5, 5, 6, 5, 5, 5, 6, 5, 5, 5,
↑ $n=12$ ↑ $n=33$ ↑ $n=54$ ↑ $n=75$ ↑ $n=96$ ↑ $n=122$ ↑ $n=143$

6, 5, 5, 5, 5, 6, 5, 5
↑ $n=164$ ↑ $n=190$

The pattern shown above can be verified directly from Tables IV and V for $n \leqslant 117$. For $118 \leqslant n \leqslant 200$, the sequence above and the entries of Table I have been verified numerically using BERNSCAN.

We observe that for $n \leqslant 115$ the exact value of $R_n = n - 4k$ can be calculated using

$$k = \left[\!\left[\frac{n-2-[\![(n-11)/21]\!]}{5} \right]\!\right],$$

where $[\![\ ]\!]$ indicates the greatest integer function.

3. Some Inequalities Involving Euler and Bernoulli Numbers

C. Jordan [9] has given inequalities for the Bernoulli numbers B_{2n} and the Euler numbers E_{2n} which yield upper and lower bounds for $|B_{2n}|$ and $|E_{2n}|$. Other estimates have been given by D. Knuth [10] and D. Leeming [12]. However, in this paper, we require more accurate estimates which are contained in the following apparently new result.

Lemma 3.1. *We have*

$$\text{(i)} \qquad 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n} < |B_{2n}| < 5\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}, \qquad n \geqslant 2 \tag{3.1}$$

$$\text{(ii)} \qquad \frac{8\sqrt{n}}{\sqrt{\pi}}\left(\frac{4n}{\pi e}\right)^{2n} < |E_{2n}| < \frac{8\sqrt{n}}{\sqrt{\pi}}\left(\frac{4n}{\pi e}\right)^{2n}\left(1+\frac{1}{12n}\right), \qquad n \geqslant 2 \tag{3.2}$$

(iii) $$2^{2n+1}\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n} < |D_{2n}| < 2^{2n+3}\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}, \qquad n \geq 2 \tag{3.3}$$

(iv) $$\frac{2n}{\pi e} < |D_{2n}|^{1/2n} < (1.705)\left(\frac{2n}{\pi e}\right), \qquad n \geq 2, \tag{3.4}$$

where D_n is given by (1.2).

Proof. Inequalities (i) and (ii) follow by applying Stirling's formula to the inequalities for $|B_{2n}|$ and $|E_{2n}|$ given in [3, p. 805] after applying the inequalities (see, e.g., [9, p. 111])

$$2\sqrt{\pi n}\left(\frac{2n}{\pi e}\right)^{2n} < (2n)! < 2\sqrt{\pi n}\left(\frac{2n}{\pi e}\right)^{2n}\left(1+\frac{1}{12n}\right). \tag{3.5}$$

Inequality (iii) follows from (3.1) and (1.2), and (iv) can be obtained from (iii) by taking $2n$th roots to obtain

$$(4\pi n)^{1/4n}\left(\frac{2n}{\pi e}\right) < |D_{2n}|^{1/2n} < 2^{1/n}(4\pi n)^{1/4n}\left(\frac{2n}{\pi e}\right), \qquad n \geq 2. \tag{3.6}$$

Now $1 < (4\pi n)^{1/4n} < \frac{3}{2}$, $n \geq 2$, and $1 < 2^{1/n}(4\pi n)^{1/4n} < 1.705$, $n \geq 3$, and a direct computation shows that (3.4) is valid for $n = 2$. ∎

4. Evaluation of $B_n(q)$, $q = m, m+\frac{1}{6}, m+\frac{1}{4}, m+\frac{1}{2}, m+\frac{3}{4}, m+\frac{5}{6}$

Let $m \geq 1$ be an integer. Using (1.3) repeatedly with x replaced successively by $x+1, ..., x+m$ we get

$$B_n(x+m) = B_n(x) + n\sum_{k=0}^{m-1}(x+k)^{n-1}. \tag{4.1}$$

We now state the following lemma, which is new and will be useful in subsequent sections (see also Inkeri [8, p. 10]).

Lemma 4.1. *Let m be a positive integer. If $B_n(y) > 0$, $0 \leq y \leq 1$, then $B_n(y+m) > 0$.*

Proof. From (4.1), $B_n(y+m) = B_n(y) + n\sum_{j=0}^{m-1}(y+j)^{n-1}$. Since $B_n(y) > 0$, the result follows. ∎

Now from Nörlund [15, p. 22], (1.2), and (1.4) we have, for $n = 1, 2, ...,$

$$B_{2n}\left(\frac{1}{6}\right) = B_{2n}\left(\frac{5}{6}\right) = \left(1-\frac{1}{2^{2n-1}}\right)\left(1-\frac{1}{3^{2n-1}}\right)\frac{B_{2n}}{2} \tag{4.2}$$

$$B_{2n}\left(\frac{1}{4}\right)=B_{2n}\left(\frac{3}{4}\right)=2^{-4n}D_{2n} \tag{4.3}$$

$$B_{2n}\left(\frac{1}{2}\right)=2^{-2n}D_{2n}. \tag{4.4}$$

Therefore,

$$B_{2n}\left(m+\frac{1}{6}\right)=\frac{B_{2n}}{2}\left(1-\frac{1}{2^{2n-1}}\right)\left(1-\frac{1}{3^{2n-1}}\right)+2n\sum_{j=0}^{m-1}\left(j+\frac{1}{6}\right)^{2n-1}, \quad m\geqslant 1 \tag{4.5}$$

$$B_{2n}\left(m+\frac{1}{4}\right)=2^{-4n-1}\left[2D_{2n}+n\sum_{j=0}^{m-1}(4j+1)^{2n-1}\right], \qquad m\geqslant 1. \tag{4.6}$$

Similarly (since $B_{2n}:=B_{2n}(0)=B_{2n}(1)$, $n\geqslant 1$),

$$B_{2n}(m)=B_{2n}+2n\sum_{j=1}^{m-1}j^{2n-1}, \qquad m\geqslant 1 \tag{4.7}$$

$$B_{2n}\left(m+\frac{1}{2}\right)=2^{-2n}\left[D_{2n}+4n\sum_{j=0}^{m-1}(2j+1)^{2n-1}\right], \qquad m\geqslant 1 \tag{4.8}$$

$$B_{2n}\left(m+\frac{3}{4}\right)=2^{-4n}\left[D_{2n}+8n\sum_{j=0}^{m-1}(4j+3)^{2n-1}\right], \qquad m\geqslant 1 \tag{4.9}$$

$$B_{2n}\left(m+\frac{5}{6}\right)=\frac{B_{2n}}{2}\left(1-\frac{1}{2^{2n-1}}\right)\left(1-\frac{1}{3^{2n-1}}\right)+2n\sum_{j=0}^{m-1}\left(j+\frac{5}{6}\right)^{2n-1}, \qquad m\geqslant 1 \tag{4.10}$$

$$B_{2n+1}\left(\frac{1}{4}\right)=-(2n+1)2^{-4n-2}E_{2n} \tag{4.11}$$

$$B_{2n+1}\left(m+\frac{1}{4}\right)=(2n+1)2^{-4n-2}\left[-E_{2n}+4\sum_{j=0}^{m-1}(4j+1)^{2n}\right], \qquad m\geqslant 1 \tag{4.12}$$

$$B_{2n+1}\left(m+\frac{1}{2}\right)=(2n+1)2^{-2n}\sum_{j=0}^{m-1}(2j+1)^{2n}>0, \qquad m\geqslant 1 \tag{4.13}$$

$$B_{2n+1}\left(m+\frac{3}{4}\right)=(2n+1)2^{-4n-2}\left[E_{2n}+4\sum_{j=0}^{m-1}(4j+3)^{2n}\right], \qquad m\geqslant 1 \tag{4.14}$$

$$B_{2n+1}(m)=(2n+1)\sum_{j=1}^{m-1}j^{2n}>0, \qquad m\geqslant 2. \tag{4.15}$$

There are no known simple closed expressions for $B_{2n+1}(m+\frac{1}{6})$ or $B_{2n+1}(m+\frac{5}{6})$.

We now consider the problem of determining the sign of $B_n(q)$ for certain prescribed values of n and q. These results are given in the following lemmas.

LEMMA 4.2. *Let m be a positive integer. Then*

$$\text{(i)} \quad B_{4n}\left(m+\frac{1}{4}\right)>0, \qquad n\geqslant 0 \tag{4.16}$$

$$\text{(ii)} \quad B_{4n+1}\left(m+\frac{1}{4}\right)<0 \qquad \textit{iff } E_{4n}>4\sum_{j=0}^{m-1}(4j+1)^{4n} \tag{4.17}$$

$$\text{(iii)} \quad B_{4n+2}\left(m+\frac{1}{4}\right)<0 \qquad \textit{iff } |D_{4n+2}|>4(4n+2)\sum_{j=0}^{m-1}(4j+1)^{4n+1} \tag{4.18}$$

$$\text{(iv)} \quad B_{4n+3}\left(m+\frac{1}{4}\right)>0, \qquad n\geqslant 0 \tag{4.19}$$

$$\text{(v)} \quad B_{4n}\left(m+\frac{3}{4}\right)>0, \qquad n\geqslant 0 \tag{4.20}$$

$$\text{(vi)} \quad B_{4n+1}\left(m+\frac{3}{4}\right)>0, \qquad n\geqslant 0 \tag{4.21}$$

$$\text{(vii)} \quad B_{4n+2}\left(m+\frac{3}{4}\right)<0 \qquad \textit{iff } |D_{4n+2}|>4(4n+2)\sum_{j=0}^{m-1}(4j+3)^{4n+1} \tag{4.22}$$

$$\text{(viii)} \quad B_{4n+3}\left(m+\frac{3}{4}\right)<0 \qquad \textit{iff } |E_{4n+2}|>4\sum_{j=0}^{m-1}(4j+3)^{4n+2}. \tag{4.23}$$

Proof. From Nörlund [15, pp. 23 and 26] and (1.2) we have $(-1)^{n+1}B_{2n}>0$, $(-1)^n E_{2n}>0$, and $(-1)^n D_{2n}>0$ for $n\geqslant 1$ (in particular $E_{4n}>0$); so using (4.12) we get (4.17). The other inequalities are proved similarly. ∎

LEMMA 4.3. *Let m be a positive integer. Then*

$$\text{(i)} \quad B_{4n}(m)<0 \qquad \textit{iff } |B_{4n}|>4n\sum_{j=1}^{m-1} j^{4n-1},\ m\geqslant 2 \tag{4.24}$$

$$\text{(ii)} \quad B_{4n+2}(m)>0, \qquad m\geqslant 1,\ n\geqslant 0 \tag{4.25}$$

(iii) $$B_{4n}\left(m+\frac{1}{2}\right)>0, \qquad m\geqslant 1,\ n\geqslant 0 \tag{4.26}$$

(iv) $$B_{4n+2}\left(m+\frac{1}{2}\right)<0 \qquad \textit{iff } |D_{4n+2}|>(8n+4)\sum_{j=0}^{m-1}(2j+1)^{4n+1},\ m\geqslant 1. \tag{4.27}$$

Proof. Inequalities (i)–(iv) follow immediately from (4.7) and (4.8) after observing that $(-1)^{n+1}B_{2n}>0$ and $(-1)^n D_{2n}>0$. ∎

Finally, we note that since $B_n(x)\to\infty$ as $n\to\infty$, inequalities (4.17), (4.23), (4.24), and (4.27) show that the largest real zero of $B_n(x)$ increases without bound as $n\to\infty$ (see also [8, p. 12]).

5. The Real Zeros of $B_{4n}(x)$ Outside the Interval [0, 1]

Since $B_{4n}(1)=B_{4n}<0$, $n\geqslant 1$, and since $B_{4n}(x)$ is a monic polynomial, we let M be the largest positive integer such that $B_{4n}(M)<0$, that is $B_{4n}(m)<0$, $m=1, 2, ..., M$ and $B_{4n}(M+1)>0$. Inkeri [8, p. 12] shows that $B_{4n}(x)$ may have either one or three zeros in the interval $(M, M+1)$ and there are no real zeros of $B_{4n}(x)$ greater than $M+1$. The occurrence of three roots in the interval $(M, M+1)$ is an irregular but persistent phenomenon as we see from Table II which lists all pairs of zeros of $B_{4n}(x)$,

TABLE II

Real Zero Pairs of $B_{4n}(x)$ in the Interval $(M+\frac{3}{4}, M+1)$, $4\leqslant 4n\leqslant 500$

$4n$	M	Zeros	
16	1	1.76	1.94
32	2	2.76	2.89
84	5	5.76	5.97
100	6	6.76	6.91
152	9	9.75	9.97
168	10	10.76	10.90
220	13	13.76	13.96
236	14	14.76	14.89
288	17	17.76	17.95
356	21	21.76	21.94
372	22	22.78	22.85
408	24	24.75	24.99
440	26	26.80	26.81
476	28	28.75	28.98
492	29	29.76	29.90

$4 \leqslant 4n \leqslant 500$ in the interval $(M+\frac{3}{4}, M+1)$. The computation of these zeros is described in Section 9.

The largest real zero of $B_{4n}(x)$ will lie anywhere in either one of the intervals $(M, M+\frac{1}{4})$ or $(M+\frac{3}{4}, M+1)$ which explains the irregular count of Inkeri. A more definitive result for the position of the real zeros of $B_{4n}(x)$ is given in Lemma 5.3. First we need the following two lemmas.

LEMMA 5.1. *Let m and n be positive integers. Then $B_{4n}(x)>0$ for $m+\frac{1}{4} \leqslant x \leqslant m+\frac{3}{4}$.*

Proof. We know $B_{4n}(\frac{1}{2})>0$ and that the only real zeros of $B_{4n}(x)$ in $(0, 1)$ are r_{4n} and $s_{4n}=1-r_{4n}$ where $\frac{1}{6}<r_{4n}<\frac{1}{4}$ (see, e.g., [14]). Therefore, $B_{4n}(x)>0$, $\frac{1}{4} \leqslant x \leqslant \frac{3}{4}$. Using Lemma 4.1, the result follows. ∎

LEMMA 5.2. *For each (fixed) integer $m \geqslant 1$, there exist positive integers j_m and k_m with $j_m \leqslant k_m$ such that*

$$\text{(i)} \quad B_{4n}\left(m+\frac{1}{6}\right)>0, \quad n<j_m; \quad B_{4n}\left(m+\frac{1}{6}\right)<0, \quad n \geqslant j_m \tag{5.1}$$

$$\text{(ii)} \quad B_{4n}\left(m+\frac{5}{6}\right)>0, \quad n<k_m; \quad B_{4n}\left(m+\frac{5}{6}\right)<0, \quad n \geqslant k_m. \tag{5.2}$$

Proof. Since $B_{4n}<0$, $n \geqslant 1$, using (4.2) we observe that for $n=1, 2, \ldots$, $B_{4n}(\frac{1}{6})=B_{4n}(\frac{5}{6})<0$. A direct calculation using (4.5) and (4.10) shows that in the case $n=1$, $B_4(m+\frac{1}{6})>0$ and $B_4(m+\frac{5}{6})>0$. Using (4.5) and (3.1), we see that, for fixed $m \geqslant 1$, $B_{4n}(m+\frac{1}{6})<0$ when n is sufficiently large, so (5.1) follows. A similar argument yields (5.2). Finally, from (4.5) and (4.10), we see that $B_{4n}(m+\frac{5}{6})>B_{4n}(m+\frac{1}{6})$ for all $m \geqslant 1$, $n \geqslant 1$, so $j_m \leqslant k_m$. ∎

LEMMA 5.3. *For $n=1, 2$, and 3, $B_{4n}(x)$ has exactly one zero in the interval $(\frac{3}{4}, \frac{5}{4})$. For $n \geqslant 4$ and m a positive integer, $B_{4n}(x)$ has either two zeros or none in the interval $(m-\frac{1}{4}, m+\frac{1}{4})$.*

Proof. We need to show that whenever $B_{4n}(x)$, $n \geqslant 4$, has one zero in the interval $(m-\frac{1}{4}, m+\frac{1}{4})$ it must have exactly one more zero in the same interval. Suppose, then, that $B_{4n}(x)$ has a zero in $(m-\frac{1}{4}, m)$. From Inkeri [8, p. 15] we have $B''_{4n}(x)=4n(4n-1)B_{4n-2}(x)>0$ for (at least) $m-\frac{1}{4}-h_n \leqslant x \leqslant m+\frac{1}{4}+h_n$ where $h_n=2^{-4n-2}\pi^{-1}$. Furthermore (see [8, p. 19]), $B'_{4n}(x)<0$ on $(m-\frac{1}{4}, m-\varepsilon_n)$ and $B'_{4n}(x)>0$ on $(m+\delta_n, m+\frac{1}{4})$ where $0<\varepsilon_n<\frac{1}{6}$, $0<\delta_n<\frac{1}{6}$, and $\varepsilon_n \to 0$, $\delta_n \to 0$ as $n \to \infty$. Therefore, $B_{4n}(m)<0$ and since from (4.16) and (4.20), we have $B_{4n}(m-\frac{1}{4})>0$ and $B_{4n}(m+\frac{1}{4})>0$, the result follows. ∎

It should be noted that because of the "crossover" phenomenon described in Section 7, a similar lemma is not possible in the case of $B_{4n+2}(x)$. However, Lemma 7.2 gives the comparable result for that case.

Finding the pairs of zeros of $B_{4n}(x)$ in $(M+\frac{3}{4}, M+1)$ involves first determining the sign of $B_{4n-1}(M+\frac{3}{4})$ and using (1.5). If $B_{4n-1}(M+\frac{3}{4})<0$, there is no *guarantee* of a pair of real zeros in $(M+\frac{3}{4}, M+1)$, however, there is that possibility. For example, $B_{99}(6.75)<0$ and we find (see Table II) that $B_{100}(x)$ has a pair of zeros in the interval (6.75, 7). On the other hand, $B_{115}(7.75)<0$ yet from Table V we see that $B_{116}(x)$ has no real zeros in the interval (7.75, 8).

The computations for Table II are done using BERNSCAN. This FORTRAN program uses (4.1) and the Fourier series expansion (see [3, p. 805])

$$B_n(y) = -\frac{2(n!)}{(2\pi)^n} \sum_{k=0}^{\infty} \cos\left(2\pi k y - \frac{1}{2}\pi n\right), \qquad 0 \leqslant y \leqslant 1, n > 1. \quad (5.3)$$

The second term in (4.1) is merely the sum of integer powers. This enables us to compute, for large values of n (up to $n=1000$), sign changes in $B_n(y+m)$ for fixed integer values of m, $m \geqslant 1$, and $0 \leqslant y \leqslant 1$. Such determinations give only very approximate values of the real zeros of $B_n(x)$ but these sign changes do enable us to accurately count the number of real zeros of $B_n(x)$ for large values of n. For example, in the case $n \equiv 0 \pmod 4$ we can use BERNSCAN to determine the exact number of real zeros (up to $n=1000$) including the cases for which the Delange estimates are not exact (see [5, p. 541]).

6. The Real Zeros of $B_{4n+1}(x)$ Outside the Interval [0, 1]

From Inkeri [8, p. 11] we have

$$\begin{aligned} &B_{4n+1}(x) \geqslant 0, && \tfrac{1}{2} \leqslant x \leqslant 1 \\ &B_{4n+1}(x) > 0, && m+\tfrac{1}{2} \leqslant x \leqslant m+1,\ m \geqslant 1 \end{aligned} \qquad (6.1)$$

and $B_{4n+1}(x)$ is convex upward on $[m, m+\frac{1}{2}]$, $m \geqslant 1$. Therefore, in the interval $[m, m+\frac{1}{2}]$, $B_{4n+1}(x)$ has either two zeros, or none. Normally, if $B_{4n+1}(x)$ has a pair of zeros in the interval $[m, m+\frac{1}{2}]$, then $B_{4n+1}(m+\frac{1}{4})<0$. (There are exceptions, however, e.g., $n=5$, $m=2$, with a pair of zeros of $B_{21}(x)$ in the subinterval $(2, \frac{9}{4})$ and with $B_{21}(\frac{9}{4})>0$.) Thus, to ensure the existence of a pair of zeros of $B_{4n+1}(x)$ in the interval $(m, m+\frac{1}{2})$, it is sufficient to determine (for fixed m) the values of n for which $B_{4n+1}(m+\frac{1}{4})<0$. These values can be computed using inequality (4.16).

The convexity of $B_{4n+1}(x)$ on $(m, m+\frac{1}{2})$, $m \geqslant 1$, enables us to obtain quite accurate upper and lower estimates for a pair of real zeros of $B_{4n+1}(x)$ lying in the interval $(m, m+\frac{1}{2})$, $m \geqslant 1$, using Newton's method or the method of false position. We describe here, in some detail, the procedure for obtaining such a pair of real zeros of $B_{4n+1}(x)$. We note that similar estimates may be obtained in the other three cases.

We observe that the properties of these polynomials dictate that Newton's method will provide a better approximation to the zeros than will the method of false position. However, for "large" values of n, the difference between the two estimates is extremely small and so give very accurate estimates for the real zeros of $B_n(x)$. When $B_{4n+1}(x)$ has two zeros in the interval $[m, m+\frac{1}{2}]$, we can obtain simple upper and lower estimates for these zeros.

(i) *Upper and lower estimates for the zero of* $B_{4n+1}(x)$ *"near"* $x=m$. We denote the real zero of $B_{4n+1}(x)$ "near" $x=m$ by $a_{n,m}$, the lower estimate by $\delta_{n,m}$, and the upper estimate by $\phi_{n,m}$. Thus we have, $\delta_{n,m} < a_{n,m} < \phi_{n,m}$ for each (fixed) m, and n sufficiently large.

Using one application of Newton's method with $x=m$ as our initial value, we obtain a lower estimate

$$\delta_{n,m} = m - B_{4n+1}(m)/B'_{4n+1}(m). \tag{6.2}$$

Using (1.5), (4.7), and (4.15), (6.2) becomes

$$\delta_{n,m} = m - \sum_{j=0}^{m-1} j^{4n} \Big/ \left[B_{4n} + 4n \sum_{j=0}^{m-1} j^{4n-1} \right], \qquad m \geqslant 1. \tag{6.3}$$

We note here that $\delta_{n,m}$ is indeed a lower estimate of $a_{n,m}$ due to the convexity of $B_{4n+1}(x)$ on $[m, m+\frac{1}{2}]$.

To obtain an upper estimate for $a_{n,m}$ we use the method of false position on the interval $[m, m+\frac{1}{4}]$, which yields

$$\varphi_{n,m} = m - \frac{1}{4} B_{4n+1}(m) \Big/ \left[B_{4n+1}\left(m+\frac{1}{4}\right) - B_{4n+1}(m) \right]. \tag{6.4}$$

Using (4.12) and (4.15) in (6.4) and simplifying, we get

$$\varphi_{n,m} = m - \sum_{j=0}^{m-1} (4j)^{4n} \Big/ \left[-E_{4n} + 4 \left(\sum_{j=0}^{m-1} (4j+1)^{4n} - \sum_{j=0}^{m-1} (4j)^{4n} \right) \right], \quad m \geqslant 1. \tag{6.5}$$

Using (1.5), (4.8), and (4.13) similar upper and lower estimates can be obtained for the zero of $B_{4n+1}(x)$ "near" $x = m+\frac{1}{2}$.

7. The Real Zeros of $B_{4n+2}(x)$ Outside the Interval [0, 1]

Inkeri [8] has pointed out that $B_{4n+2}(x) > 0$, $m = 0, \pm 1, \pm 2, \ldots$, and $B_{4n+2}(x)$ has at most one zero in the interval $[m + \frac{1}{2}, m + 1]$, and either two zeros or none in the interval $(m, m + 1)$. Furthermore, if $B_{4n+2}(m + \frac{1}{2}) > 0$ for some value of m, then every real zero of $B_{4n+2}(x)$ is less than $m + 1$. Thus if $m = M$ is the largest integer such that $B_{4n+2}(m + \frac{1}{2}) < 0$, then there are no zeros of $B_{4n+2}(x)$ greater than $M + 2$. In what follows we assume m is an integer such that $1 \leqslant m \leqslant M$.

The "crossover" phenomenon. It is well known (see, e.g., [13]) that the pair of zeros of $B_{4n+2}(x)$ in the interval [0, 1] converge *monotonically* to $\frac{1}{4}$ and $\frac{3}{4}$ as $n \to \infty$. It is easily shown that this monotonic behavior is also exhibited by the pair of zeros of $B_{4n+2}(x)$ in the interval [1, 2], which converge $(n \to \infty)$ to $\frac{5}{4}$ and $\frac{7}{4}$. The behavior of the pair of real zeros of $B_{4n+2}(x)$ in the intervals $[m, m + 1]$, $m \geqslant 2$, is not monotonic, however, as it was in the case $m = 0$ and $m = 1$. We describe here this new concept for the interval [2, 3] and then give the general case in Lemma 7.1.

Let $p_{n,2}$ denote the real zero of $B_{4n+2}(x)$ "near" $x = 2.25$. Table III shows that for $n = 6, \ldots, 10$, $p_{n,2} > 2.25$ but for $n \geqslant 11$, $p_{n,2} < 2.25$. Whereas Inkeri [8] shows that $\{p_{n,2}\}$ is an increasing sequence for $n \geqslant n_0$ (in this case, $n_0 = 11$), the behavior of the sequence of zeros $\{p_{n,2}\}$ for $n \leqslant 10$ is *monotonic decreasing*. In addition, if we let $q_{n,2}$ denote the real of $B_{4n+2}(x)$ "near" $x = 2.75$, we have for $n \leqslant 14$, $q_{n,2} < 2.75$ and $\{q_{n,2}\}$ is *monotonic increasing*, while for $n \geqslant 15$, $q_{n,2} > 2.75$ and $\{q_{n,2}\}$ is *monotonic decreasing*, as $n \to \infty$, to 2.75.

Inkeri's work [8, Theorem 1, p. 4] predicts the asymptotic behavior of the real zeros of $B_{4n+2}(x)$ on the interval $[m, m + 1]$ but he does not mention the crossover phenomenon for $m \geqslant 2$ described above. We now formalize this description and give a more precise statement than that of Inkeri for the position of the real zeros of $B_{4n+2}(x)$.

TABLE III

m	h_m	l_m
2	11	15
3	19	23
4	28	32
5	36	40
6	45	49
7	53	57
8	62	66
9	70	74
10	79	83

LEMMA 7.1. *For each integer $m \geqslant 2$, there exist positive integers h_m and l_m such that*

(i) $B_{4n+2}\left(m+\frac{1}{4}\right)>0, \quad n<h_m; \qquad B_{4n+2}\left(m+\frac{1}{4}\right)<0, \quad n \geqslant h_m$

(ii) $B_{4n+2}\left(m+\frac{3}{4}\right)>0, \quad n<l_m; \qquad B_{4n+2}\left(m+\frac{3}{4}\right)<0, \quad n \geqslant l_m.$

Proof. In the case $m=2$, we have obtained directly by computation $h_2=11$ and $l_2=15$. Setting $n=3$ in (4.6) and using (1.2) we have

$$B_6\left(m+\frac{1}{4}\right)=2^{-13}\left[-\frac{31}{21}+3\sum_{j=0}^{m-1}(4j+1)^5\right]>0, \qquad m \geqslant 2.$$

However, from (1.2) and (3.1) we have

$$D_{4n+2}=2(1-2^{4n+1})B_{4n+2}<0 \tag{7.1}$$

and

$$2^{4n+2}B_{4n+2} \sim 4\sqrt{(2n+1)\pi}\left(\frac{4n+2}{2\pi e}\right)^{4n+2} \qquad (n \to \infty). \tag{7.2}$$

Therefore, in (4.6), the (negative) sign of D_{4n+2} will determine the sign of $B_{4n+2}(m+\frac{1}{4})$ for sufficiently large n, hence there exists a smallest integer h_m such that $B_{4n+2}(m+\frac{1}{4})<0$, $n \geqslant h_m$. The proof of (ii) is similar. ∎

We observe from (4.6) and (4.9) that $l_m \geqslant h_m$. Some specific values of l_m and h_m are given in Table III. We note from Table III that for $2 \leqslant m \leqslant 10$, $l_m=h_m+4$ although it is not known whether or not this equality holds for larger values of m.

From Table III we can determine three different forms, or stages, for the position of the pair of real zeros of $B_{4n+2}(x)$, $n \geqslant n_m$, in the interval $[m, m+1]$, $m \geqslant 2$. These three stages are shown in Fig. 1.

In spite of the "crossover" phenomenon, it is still possible to obtain accurate estimates for the two zeros of $B_{4n+2}(x)$ in the interval $(m, m+1)$ using Newton's method. In this case, the advantage over Inkeri's results is that our estimates will follow the "crossover," and so are good for all values of $n \geqslant h_m$ where h_m is as defined in Lemma 7.1. Furthermore, these estimates exhibit the same order of accuracy as Inkeri's estimates, namely $O(2^{-4n})$, as $n \to \infty$. This is easily shown using (4.3), (4.6), (4.9), (4.11), (1.5), and the estimates of Section 3.

$B_{4n+2}(x)$ on the interval $[m, m+1]$, $m \geq 2$

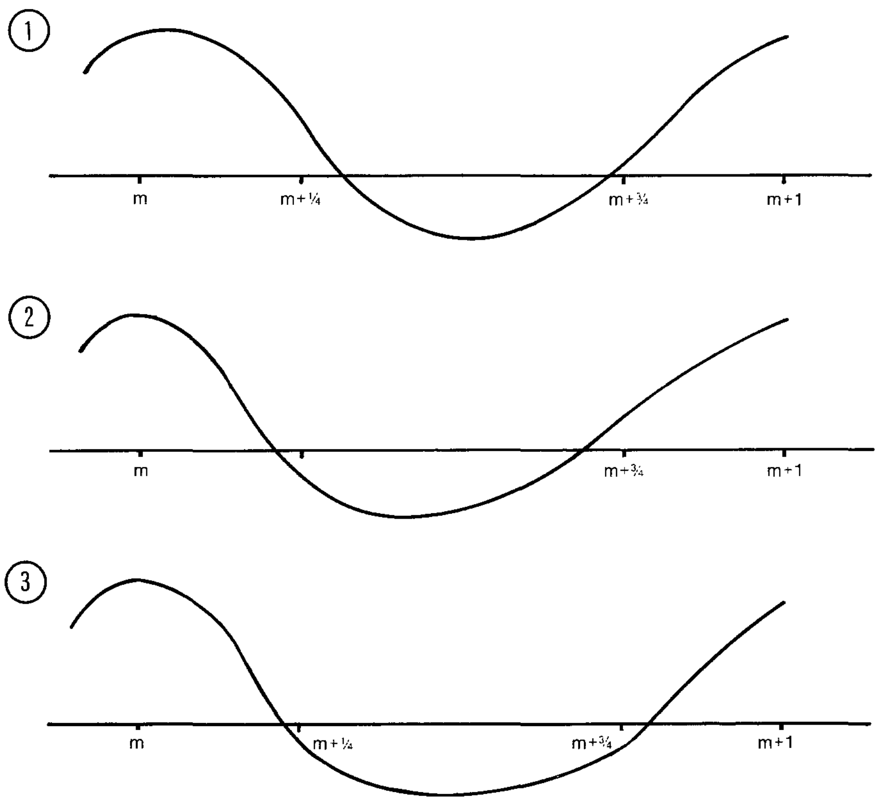

FIG. 1. (1) $n < h_m$; (2) $h_m \leqslant n < l_m$; (3) $n \geqslant l_m$.

8. The Real Zeros of $B_{4n+3}(x)$ Outside the Interval [0, 1]

In this case, Inkeri [8] shows that

$$\begin{aligned} &B_{4n+3}(x) \geqslant 0, \qquad 1 \leqslant x \leqslant \frac{3}{2} \\ &B_{4n+3}(x) > 0, \qquad m \leqslant x \leqslant m + \frac{1}{2}, \; m \geqslant 2 \end{aligned} \tag{8.1}$$

and in the intervals $(m+\frac{1}{2}, m+1)$, $B_{4n+3}(x)$ has, at most, two zeros. Usually if $B_{4n+3}(x)$ has a pair of zeros in the interval $(m+\frac{1}{2}, m+1)$, then $B_{4n+3}(m+\frac{3}{4}) < 0$. The case $n=2$, $m=1$, is an exception, however, with

$B_{11}(\frac{7}{4})>0$. In a similar fashion to the $B_{4n+1}(x)$ case, we can determine the smallest value of n, say n_m, such that $B_{4n+3}(m+\frac{3}{4})<0$ for a fixed value of m. Then for $n \geqslant n_m$, $B_{4n+3}(x)$ will have exactly two zeros in the interval $(m+\frac{1}{2}, m+1)$, one in $(m+\frac{1}{2}, m+\frac{3}{4})$ and the other in $(m+\frac{3}{4}, m+1)$.

(i) *Upper and lower estimates for the zeros of $B_{4n+3}(x)$ "near" $x=m+\frac{1}{2}$.* We denote the real zero of $B_{4n+3}(x)$ "near" $x=m+\frac{1}{2}$ by $g_{n,m}$, our lower estimate by $\alpha_{n,m}$, and our upper estimate by $\beta_{n,m}$.

To obtain the lower estimate $\alpha_{n,m}$ for $g_{n,m}$ we use one application of Newton's method with initial value $x=m+\frac{1}{2}$, (4.8), (4.13), and (1.5) which yields

$$\alpha_{n,m}=\left(m+\frac{1}{2}\right)-\frac{\sum_{j=1}^{m}(2j-1)^{4n+2}}{[D_{4n+2}+(8n+4)\sum_{j=1}^{m}(2j-1)^{4n+1}]}, \qquad m \geqslant 1. \tag{8.2}$$

To obtain the upper estimate $\beta_{n,m}$ for $g_{n,m}$ we use the method of false position on $[m+\frac{1}{2}, m+\frac{3}{4}]$ and obtain

$$\beta_{n,m}=\left(m+\frac{1}{2}\right)-\frac{\sum_{j=1}^{m}(4j-2)^{4n+2}}{[E_{4n+2}+4(\sum_{j=1}^{m}(4j-1)^{4n+2}-\sum_{j=1}^{m}(4j-2)^{4n+2})]}. \tag{8.3}$$

(ii) *Comparison of our estimates with Inkeri's estimates.* Let m and n be positive integers with n sufficiently large so that $B_{4n+3}(x)$ has a pair of zeros in the interval $[m+\frac{1}{2}, m+1]$. Let $g_{n,m}$ denote the real zero of $B_{4n+3}(x)$ "near" $x=m+\frac{1}{2}$, and let $\eta_{n,m}$ and $\mu_{n,m}$ be the lower and upper estimates, respectively, for $g_{n,m}$ given by Inkeri [8, p. 19]. Let $\alpha_{n,m}$ and $\beta_{n,m}$ be as given in (8.2) and (8.3). Then we have the following theorem.

THEOREM 8.1. *For n sufficiently large, $m=1, 2, \ldots$,*

$$m+\frac{1}{2}<\eta_{n,m}<\alpha_{n,m}<g_{n,m}<\beta_{n,m}<\mu_{n,m} \tag{8.4}$$

and

$$\lim_{n\to\infty}\mu_{n,m}=m+\frac{1}{2}. \tag{8.5}$$

Proof. Inkeri [8, p. 18] has shown that

$$B_{4n+3}\left(m+\frac{1}{2}+H_{n,m}\right)=B_{4n+3}(\eta_{n,m})<0$$

$$B_{4n+3}\left(m+\frac{1}{2}+H_{n+1,m}\right)=B_{4n+3}(\mu_{n,m})>0,$$

where

$$H_{n,m}=\frac{[(2m-1)\pi]^{4n+1}}{2(4n+1)!}. \tag{8.6}$$

From (8.2) and (8.3) we can define $s_{n,m}$ and $t_{n,m}$ by

$$\begin{aligned} \alpha_{n,m}&=m+\frac{1}{2}+s_{n,m} \\ \beta_{n,m}&=m+\frac{1}{2}+t_{n,m}, \end{aligned} \tag{8.7}$$

respectively. Using Stirling's formula and taking $(4n+2)$th roots, we find

$$|H_{n,m}|^{1/(4n+2)} \sim \frac{(2m-1)\pi e}{4n+1}$$

and

$$|H_{n+1,m}|^{1/(4n+2)} \sim \frac{(2m-1)\pi e}{4n+3} \qquad (n\to\infty), \tag{8.8}$$

also

$$|s_{n,m}|^{1/(4n+2)} \sim \frac{(2m-1)\pi e}{4n+2}$$

and

$$|t_{n,m}|^{1/(4n+2)} \sim \frac{(2m-1)\pi e}{4n+2} \qquad (n\to\infty). \tag{8.9}$$

Comparing the asymptotic estimates in (8.8) and (8.9) yields (8.4). To obtain (8.5), observe that $\mu_{n,m}=m+\frac{1}{2}+H_{n+1,m}$ and, by (8.6), $H_{n+1,m}\to 0$ as $n\to\infty$. ▮

Theorem 8.1 shows that our upper and lower bounds for $g_{n,m}$ are asymptotically better than those of Inkeri (see (8.4)). Numerical evidence shows that, for a given $m\geqslant 1$, our bounds are always better than those of Inkeri, even for "small' values of n. The estimates for the real zero of $B_{4n+3}(x)$ "near" $x=m$, $m\geqslant 1$, can be obtained in a similar way.

9. Computation of the Real Zeros of $B_n(x)$, $3\leqslant n\leqslant 117$

The computations of the real and complex zeros of $B_n(x)$, $3\leqslant n\leqslant 83$, were done on an IBM 3083 at the University of Victoria using a NAG FORTRAN library routine CO2AEF. The routine, which we modified to allow computations in quadruple precision, finds the zeros of a real polynomial using a method of Grant and Hitchins [7]. Successive zeros are found to within limiting machine precision, in this case approximately 32

TABLE IV

The Positive Real Zeros of $B_n(x)$, $3 \leqslant n \leqslant 83$

n	r	Zero
3	1	0.50000 00000 00000
3	2	1.00000 00000 00000
4	1	0.24033 51888 20385
4	2	0.75966 48111 79614
4	3	1.15770 37219 25804
5	1	0.50000 00000 00000
5	2	1.00000 00000 00000
5	3	1.26376 26158 25973
6	1	0.24754 07162 43673
6	2	0.75245 92837 56326
7	1	0.50000 00000 00000
7	2	1.00000 00000 00000
8	1	0.24938 03839 22670
8	2	0.75061 96160 77329
8	3	1.24721 52939 42496
9	1	0.50000 00000 00000
9	2	1.00000 00000 00000
9	3	1.44910 60039 63995
10	1	0.24984 47169 92154
10	2	0.75015 52830 07845
10	3	1.24992 47242 76254
10	4	1.57397 13186 63037
11	1	0.50000 00000 00000
11	2	1.00000 00000 00000
11	3	1.51868 24322 69871
11	4	1.61803 39887 49894
12	1	0.24996 11530 13877
12	2	0.75003 88469 86122
12	3	1.24995 93567 97830
13	1	0.50000 00000 00000
13	2	1.00000 00000 00000
13	3	1.49905 65978 32064
14	1	0.24999 02865 38064
14	2	0.75000 97134 61935
14	3	1.24999 03149 84616
14	4	1.72213 12800 03333
15	1	0.50000 00000 00000
15	2	1.00000 00000 00000
15	3	1.50005 23990 54264
15	4	1.86771 07183 14650
16	1	0.24999 75715 25127
16	2	0.75000 24284 74872
16	3	1.24999 75711 90774
16	4	1.75534 05925 20679
16	5	1.94308 70646 60557
17	1	0.50000 00000 00000
17	2	1.00000 00000 00000
17	3	1.49999 78485 80546
18	1	0.24999 93928 74383
18	2	0.75000 06071 25616
18	3	1.24999 93928 77416
18	4	1.74961 22926 43329
19	1	0.50000 00000 00000
19	2	1.00000 00000 00000
19	3	1.50000 00693 95200
19	4	1.98589 46768 43007
20	1	0.24999 98482 18163
20	2	0.75000 01517 81836
20	3	1.24999 98482 18141
20	4	1.75002 56047 59155
20	5	2.15349 60067 93051
21	1	0.50000 00000 00000
21	2	1.00000 00000 00000
21	3	1.49999 99981 97633
21	4	2.00196 56681 44463
21	5	2.24815 17929 04584
22	1	0.24999 99620 54513
22	2	0.75000 00379 45486
22	3	1.24999 99620 54513
22	4	1.74999 86930 95115
23	1	0.50000 00000 00000
23	2	1.00000 00000 00000
23	3	1.50000 00000 38503
23	4	1.99983 90752 62074
24	1	0.24999 99905 13626
24	2	0.75000 00094 86373
24	3	1.24999 99905 13626
24	4	1.75000 00685 09541
24	5	2.24338 31298 60960
25	1	0.50000 00000 00000
25	2	1.00000 00000 00000
25	3	1.49999 99999 99311
25	4	2.00001 15531 80755
25	5	2.43305 09240 16126
26	1	0.24999 99976 28406
26	2	0.75000 00023 71593
26	3	1.24999 99976 28406
26	4	1.75000 00001 87090
26	5	2.25078 04563 58050
26	6	2.54476 86368 29637
27	1	0.50000 00000 00000
27	2	1.00000 00000 00000
27	3	1.50000 00000 00010
27	4	1.99999 92985 12999
28	1	0.24999 99994 07101
28	2	0.75000 00005 92898
28	3	1.24999 99994 07101
28	4	1.75000 00006 62001
28	5	2.24993 25805 22957
29	1	0.50000 00000 00000
29	2	1.00000 00000 00000
29	3	1.49999 99999 99999
29	4	2.00000 00366 32447
29	5	2.49704 55873 53229
30	1	0.24999 99998 51775
30	2	0.75000 00001 48224
30	3	1.24999 99998 51775
30	4	1.75000 00001 46334
30	5	2.25000 51295 21772
30	6	2.70663 65255 47763

Table continued

TABLE IV—*Continued*

n	r	Zero
31	1	0.50000 00000 00000
31	2	1.00000 00000 00000
31	3	1.50000 00000 00000
31	4	1.99999 99983 37713
31	5	2.50032 07968 94052
31	6	2.83579 17233 96342
32	1	0.24999 99999 62943
32	2	0.75000 00000 37056
32	3	1.24999 99999 62943
32	4	1.75000 00000 37101
32	5	2.24999 96597 65619
32	6	2.76505 48479 10860
32	7	2.88809 16455 75994
33	1	0.50000 00000 00000
33	2	1.00000 00000 00000
33	3	1.50000 00000 00000
33	4	2.00000 00000 66153
33	5	2.49997 14760 24074
34	1	0.24999 99999 90735
34	2	0.75000 00000 09264
34	3	1.24999 99999 90735
34	4	1.75000 00000 09263
34	5	2.25000 00198 63150
34	6	2.74871 19640 20441
35	1	0.50000 00000 00000
35	2	1.00000 00000 00000
35	3	1.50000 00000 00000
35	4	1.99999 99999 97672
35	5	2.50000 22596 74978
35	6	2.97417 91374 17967
36	1	0.24999 99999 97683
36	2	0.75000 00000 02316
36	3	1.24999 99999 97683
36	4	1.75000 00000 02316
36	5	2.24999 99989 67574
36	6	2.75013 44411 46607
36	7	3.12222 64528 11225
37	1	0.50000 00000 00000
37	2	1.00000 00000 00000
37	3	1.50000 00000 00000

n	r	Zero
37	4	2.00000 00000 00072
37	5	2.49999 98407 08067
37	6	3.00553 73914 09667
37	7	3.19837 28261 11086
38	1	0.24999 99999 99420
38	2	0.75000 00000 00579
38	3	1.24999 99999 99420
38	4	1.75000 00000 00579
38	5	2.25000 00000 47125
38	6	2.74998 78329 77230
39	1	0.50000 00000 00000
39	2	1.00000 00000 00000
39	3	1.50000 00000 00000
39	4	1.99999 99999 99997
39	5	2.50000 00100 63579
39	6	2.99944 30339 09457
40	1	0.24999 99999 99855
40	2	0.75000 00000 00144
40	3	1.24999 99999 99855
40	4	1.75000 00000 00144
40	5	2.24999 99999 97869
40	6	2.75000 09928 72232
40	7	3.23594 19492 16971
41	1	0.50000 00000 00000
41	2	1.00000 00000 00000
41	3	1.50000 00000 00000
41	4	2.00000 00000 00000
41	5	2.49999 99994 26979
41	6	3.00005 70446 58283
41	7	3.40433 14518 78072
42	1	0.24999 99999 99963
42	2	0.75000 00000 00036
42	3	1.24999 99999 99963
42	4	1.75000 00000 00036
42	5	2.25000 00000 00038
42	6	2.74999 99268 06128
42	7	3.25227 73355 86538
42	8	3.49808 18053 93144
43	1	0.50000 00000 00000
43	2	1.00000 00000 00000
43	3	1.50000 00000 00000

n	r	Zero
43	4	1.99999 99999 99999
43	5	2.50000 00000 29558
43	6	2.99999 47753 35253
44	1	0.24999 99999 99990
44	2	0.75000 00000 00009
44	3	1.24999 99999 99990
44	4	1.75000 00000 00009
44	5	2.24999 99999 99988
44	6	2.75000 00048 99983
44	7	3.24975 93374 31681
45	1	0.50000 00000 00000
45	2	1.00000 00000 00000
45	3	1.50000 00000 00000
45	4	2.00000 00000 00000
45	5	2.49999 99999 98612
45	6	3.00000 04361 22849
45	7	3.49292 63499 47099
46	1	0.24999 99999 99997
46	2	0.75000 00000 00002
46	3	1.24999 99999 99997
46	4	1.75000 00000 00002
46	5	2.24999 99999 99997
46	6	2.74999 99997 00800
46	7	3.25002 44161 73988
46	8	3.68194 85793 23099
47	1	0.50000 00000 00000
47	2	1.00000 00000 00000
47	3	1.50000 00000 00000
47	4	2.00000 00000 00000
47	5	2.50000 00000 00059
47	6	2.99999 99667 29929
47	7	3.50097 20384 00345
47	8	3.79196 65329 69238
48	1	0.24999 99999 99999
48	2	0.75000 00000 00000
48	3	1.24999 99999 99999
48	4	1.75000 00000 00000
48	5	2.24999 99999 99999
48	6	2.75000 00000 16732
48	7	3.24999 77441 29564

Table continued

TABLE IV—*Continued*

n	r	Zero
49	1	0.50000 00000 00000
49	2	1.00000 00000 00000
49	3	1.50000 00000 00000
49	4	2.00000 00000 00000
49	5	2.49999 99999 99997
49	6	3.00000 00023 28813
49	7	3.49989 57811 43804
50	1	0.24999 99999 99999
50	2	0.75000 00000 00000
50	3	1.24999 99999 99999
50	4	1.75000 00000 00000
50	5	2.24999 99999 99999
50	6	2.74999 99999 99140
50	7	3.25000 01917 00782
50	8	3.74663 52008 84278
51	1	0.50000 00000 00000
51	2	1.00000 00000 00000
51	3	1.50000 00000 00000
51	4	2.00000 00000 00000
51	5	2.50000 00000 00000
51	6	2.99999 99998 49897
51	7	3.50001 05191 42819
51	8	3.95467 96201 45610
52	1	0.24999 99999 99999
52	2	0.75000 00000 00000
52	3	1.24999 99999 99999
52	4	1.75000 00000 00000
52	5	2.24999 99999 99999
52	6	2.75000 00000 00040
52	7	3.24999 99849 75275
52	8	3.75042 15617 63847
52	9	4.08148 07155 48561
53	1	0.50000 00000 00000
53	2	1.00000 00000 00000
53	3	1.50000 00000 00000
53	4	2.00000 00000 00000
53	5	2.49999 99999 99999
53	6	3.00000 00000 08937
53	7	3.49999 90215 32558
53	8	4.01733 69202 27736
53	9	4.12968 77448 46078
54	1	0.24999 99999 99999
54	2	0.75000 00000 00000
54	3	1.24999 99999 99999
54	4	1.75000 00000 00000
54	5	2.24999 99999 99999
54	6	2.74999 99999 99998
54	7	3.25000 00010 89562
54	8	3.74995 47276 94633
55	1	0.50000 00000 00000
55	2	1.00000 00000 00000
55	3	1.50000 00000 00000
55	4	2.00000 00000 00000
55	5	2.50000 00000 00000
55	6	2.99999 99999 99506
55	7	3.50000 00843 57972
55	8	3.99845 18466 60155
56	1	0.24999 99999 99999
56	2	0.75000 00000 00000
56	3	1.24999 99999 99999
56	4	1.75000 00000 00000
56	5	2.24999 99999 99999
56	6	2.75000 00000 00000
56	7	3.24999 99999 26680
56	8	3.75000 45553 29337
56	9	4.22212 40821 28745
57	1	0.50000 00000 00000
57	2	1.00000 00000 00000
57	3	1.50000 00000 00000
57	4	2.00000 00000 00000
57	5	2.50000 00000 00000
57	6	3.00000 00000 00025
57	7	3.49999 99932 42060
57	8	4.00018 42714 55922
57	9	4.36713 25211 29057
58	1	0 24999 99999 99999
58	2	0.75000 00000 00000
58	3	1.24999 99999 99999
58	4	1.75000 00000 00000
58	5	2.24999 99999 99999
58	6	2.74999 99999 99999
58	7	3.25000 00000 04590
58	8	3.74999 95739 71532
58	9	4.25652 53700 66502
58	10	4.43981 82178 18564
59	1	0.50000 00000 00000
59	2	1.00000 00000 00000
59	3	1.50000 00000 00000
59	4	2.00000 00000 00000
59	5	2.50000 00000 00000
59	6	2.99999 99999 99998
59	7	3.50000 00005 04372
59	8	3.99998 02713 25522
60	1	0.25000 00000 00000
60	2	0.75000 00000 00000
60	3	1.25000 00000 00000
60	4	1.75000 00000 00000
60	5	2.25000 00000 00000
60	6	2.75000 00000 00000
60	7	3.24999 99999 99731
60	8	3.75000 00371 69602
60	9	4.24929 99031 90521
61	1	0.50000 00000 00000
61	2	1.00000 00000 00000
61	3	1.50000 00000 00000
61	4	2.00000 00000 00000
61	5	2.50000 00000 00000
61	6	3.00000 00000 00000
61	7	3.49999 99999 64844
61	8	4.00000 19809 78427
61	9	4.48426 21719 74881
62	1	0.25000 00000 00000
62	2	0.75000 00000 00000
62	3	1.25000 00000 00000
62	4	1.75000 00000 00000
62	5	2.25000 00000 00000
62	6	2.75000 00000 00000
62	7	3.25000 00000 00014
62	8	3.74999 99969 67983
62	9	4.25008 09062 02423
62	10	4.64897 51046 50083

Table continued

TABLE IV—*Continued*

n	r	Zero
63	1	0.50000 00000 00000
63	2	1.00000 00000 00000
63	3	1.50000 00000 00000
63	4	2.00000 00000 00000
63	5	2.50000 00000 00000
63	6	2.99999 99999 99999
63	7	3.50000 00000 02293
63	8	3.99999 98139 02049
63	9	4.50276 52294 85166
63	10	4.73924 48816 94695
64	1	0.25000 00000 00000
64	2	0.75000 00000 00000
64	3	1.25000 00000 00000
64	4	1.75000 00000 00000
64	5	2.25000 00000 00000
64	6	2.75000 00000 00000
64	7	3.24999 99999 99999
64	8	3.75000 00002 31752
64	9	4.24999 13772 86866
65	1	0.50000 00000 00000
65	2	1.00000 00000 00000
65	3	1.50000 00000 00000
65	4	2.00000 00000 00000
65	5	2.50000 00000 00000
65	6	3.00000 00000 00000
65	7	3.49999 99999 99859
65	8	4.00000 00163 99293
65	9	4.49968 60025 53823
66	1	0.25000 00000 00000
66	2	0.75000 00000 00000
66	3	1.25000 00000 00000
66	4	1.75000 00000 00000
66	5	2.25000 00000 00000
66	6	2.75000 00000 00000
66	7	3.25000 00000 00000
66	8	3.74999 99999 83367
66	9	4.25000 08644 84410
66	10	4.74178 43304 19586
67	1	0.50000 00000 00000
67	2	1.00000 00000 00000
67	3	1.50000 00000 00000
67	4	2.00000 00000 00000
67	5	2.50000 00000 00000
67	6	3.00000 00000 00000
67	7	3.50000 00000 00008
67	8	3.99999 99986 41780
67	9	4.50003 56245 84294
67	10	4.92676 59145 52273
68	1	0.25000 00000 00000
68	2	0.75000 00000 00000
68	3	1.25000 00000 00000
68	4	1.75000 00000 00000
68	5	2.25000 00000 00000
68	6	2.75000 00000 00000
68	7	3.24999 99999 99999
68	8	3.75000 00000 01122
68	9	4.24999 99184 81412
68	10	4.75121 50606 52365
68	11	5.03303 25834 65794
69	1	0.50000 00000 00000
69	2	1.00000 00000 00000
69	3	1.50000 00000 00000
69	4	2.00000 00000 00000
69	5	2.50000 00000 00000
69	6	3.00000 00000 00000
69	7	3.49999 99999 99999
69	8	4.00000 00001 05922
69	9	4.49999 62213 35962
70	1	0.75000 00000 00000
70	2	0.25000 00000 00000
70	3	1.25000 00000 00000
70	4	1.75000 00000 00000
70	5	2.25000 00000 00000
70	6	2.75000 00000 00000
70	7	3.25000 00000 00000
70	8	3.74999 99999 99928
70	9	4.25000 00072 44791
70	10	4.74985 96830 42692
71	1	0.50000 00000 00000
71	2	1.00000 00000 00000
71	3	1.50000 00000 00000
71	4	2.00000 00000 00000
71	5	2.50000 00000 00000
71	6	3.00000 00000 00000
71	7	3.50000 00000 00000
71	8	3.99999 99999 92208
71	9	4.50000 03783 74748
71	10	4.99595 73899 14472
72	1	0.25000 00000 00000
72	2	0.75000 00000 00000
72	3	1.25000 00000 00000
72	4	1.75000 00000 00000
72	5	2.25000 00000 00000
72	6	2.75000 00000 00000
72	7	3.25000 00000 00000
72	8	3.75000 00000 00004
72	9	4.24999 99993 92150
72	10	4.75001 57190 58665
72	11	5.20006 51354 83756
73	1	0.50000 00000 00000
73	2	1.00000 00000 00000
73	3	1.50000 00000 00000
73	4	2.00000 00000 00000
73	5	2.50000 00000 00000
73	6	3.00000 00000 00000
73	7	3.49999 99999 99999
73	8	4.00000 00000 00541
73	9	4.49999 99642 04905
73	10	5.00054 13478 94124
73	11	5.32264 75701 99225
74	1	0.25000 00000 00000
74	2	0.75000 00000 00000
74	3	1.25000 00000 00000
74	4	1.75000 00000 00000
74	5	2.25000 00000 00000
74	6	2.75000 00000 00000
74	7	3.25000 00000 00000
74	8	3.74999 99999 99999
74	9	4.25000 00000 48224
74	10	4.74999 83402 21382
74	11	5.27291 79676 22995
74	12	5.36198 19977 85887

Table continued

TABLE IV—*Continued*

n	r	Zero	n	r	Zero	n	r	Zero
75	1	0.50000 00000 00000	78	3	1.25000 00000 00000	81	1	0.50000 00000 00000
75	2	1.00000 00000 00000	78	4	1.75000 00000 00000	81	2	1.00000 00000 00000
75	3	1.50000 00000 00000	78	5	2.25000 00000 00000	81	3	1.50000 00000 00000
75	4	2.00000 00000 00000	78	6	2.75000 00000 00000	81	4	2.00000 00000 00000
75	5	2.50000 00000 00000	78	7	3.25000 00000 00000	81	5	2.50000 00000 00000
75	6	3.00000 00000 00000	78	8	3.74999 99999 99999	81	6	3.00000 00000 00000
75	7	3.500000 00000 00000	78	9	4.25000 00000 00258	81	7	3.50000 00000 00000
75	8	3.99999 99999 99964	78	10	4.74999 99842 48885	81	8	4.00000 00000 00000
75	9	4.50000 00032 04535	78	11	5.25024 25268 64091	81	9	4.49999 99999 98324
75	10	4.99993 73867 70419	78	12	5.60860 15242 68262	81	10	5.00000 00730 18295
						81	11	5.49911 12641 70277
76	1	0.25000 00000 00000	79	1	0.50000 00000 00000	82	1	0.25000 00000 00000
76	2	0.75000 00000 00000	79	2	1.00000 00000 00000	82	2	0.75000 00000 00000
76	3	1.25000 00000 00000	79	3	1.50000 00000 00000	82	3	1.25000 00000 00000
76	4	1.75000 00000 00000	79	4	2.00000 00000 00000	82	4	1.75000 00000 00000
76	5	2.25000 00000 00000	79	5	2.50000 00000 00000	82	5	2.25000 00000 00000
76	6	2.75000 00000 00000	79	6	3.00000 00000 00000	82	6	2.75000 00000 00000
76	7	3.25000 00000 00000	79	7	3.50000 00000 00000	82	7	3.25000 00000 00000
76	8	3.75000 00000 00000	79	8	3.99999 99999 99999	82	8	3.75000 00000 00000
76	9	4.24999 99999 96376	79	9	4.50000 00000 21892	82	9	4.25000 00000 00001
76	10	4.75000 01660 33196	79	10	4.99999 92694 29336	82	10	4.74999 99998 78420
76	11	5.24808 42322 57486	79	11	5.50823 56620 59260	82	11	5.25000 30750 72695
			79	12	5.67597 96302 91672	82	12	5.73159 59090 22443
77	1	0.50000 00000 00000	80	1	0.25000 00000 00000	83	1	0.50000 00000 00000
77	2	1.00000 00000 00000	80	2	0.75000 00000 00000	83	2	1.00000 00000 00000
77	3	1.50000 00000 00000	80	3	1.25000 00000 00000	83	3	1.50000 00000 00000
77	4	2.00000 00000 00000	80	4	1.75000 00000 00000	83	4	2.00000 00000 00000
77	5	2.50000 00000 00000	80	5	2.25000 00000 00000	83	5	2.50000 00000 00000
77	6	3.00000 00000 00000	80	6	2.75000 00000 00000	83	6	3.00000 00000 00000
77	7	3.50000 00000 00000	80	7	3.25000 00000 00000	83	7	3.50000 00000 00000
77	8	4.00000 00000 00002	80	8	3.75000 00000 00000	83	8	4.00000 00000 00000
77	9	4.49999 99997 28114	80	9	4.24999 99999 99982	83	9	4.50000 00000 00121
77	10	5.00000 69475 44747	80	10	4.75000 00014 19096	83	10	4.99999 99930 55965
77	11	5.46840 28607 41961	80	11	5.24997 20718 68060	83	11	5.50010 88881 17443
78	1	0.25000 00000 00000				83	12	5.89096 49637 14264
78	2	0.75000 00000 00000						

decimal places, using a FORTVS compiler. A composite deflation technique is used throughout.

A check of the computations of the real and complex zeros of $B_n(x)$ up to $n = 42$ was made using a listing of Leon Lander and D. H. Lehmer provided by J. Brillhart [2]. For higher values of n, the lower and upper bounds provided by D. H. Lehmer [13] for $m = 1$, and by Inkeri [8] and our own estimates for $m \geqslant 1$ were used to verify that the NAG FORTRAN

TABLE V

The Positive Real Zeros of $B_n(x)$, $84 \leqslant n \leqslant 117$

n	r	Root
84	1	0.25000 00000 00
84	2	0.75000 00000 00
84	3	1.25000 00000 00
84	4	1.75000 00000 00
84	5	2.25000 00000 00
84	6	2.75000 00000 00
84	7	3.25000 00000 00
84	8	3.75000 00000 00
84	9	4.25000 00000 00
84	10	4.75000 00000 10
84	11	5.24999 96778 39
84	12	5.75349 89846 16
84	13	5.97658 50377 37
85	1	0.50000 00000 00
85	2	1.00000 00000 00
85	3	1.50000 00000 00
85	4	2.00000 00000 00
85	5	2.50000 00000 00
85	6	3.00000 00000 00
85	7	3.50000 00000 00
85	8	4.00000 00000 00
85	9	4.50000 00000 00
85	10	5.00000 00006 29
85	11	5.49998 75420 91
86	1	0.25000 00000 00
86	2	0.75000 00000 00
86	3	1.25000 00000 00
86	4	1.75000 00000 00
86	5	2.25000 00000 00
86	6	2.75000 00000 00
86	7	3.25000 00000 00
86	8	3.75000 00000 00
86	9	4.25000 00000 00
86	10	4.74999 99999 99
86	11	5.25000 00321 75
86	12	5.74959 23895 63
87	1	0.50000 00000 00
87	2	1.00000 00000 00
87	3	1.50000 00000 00
87	4	2.00000 00000 00
87	5	2.50000 00000 00
87	6	3.00000 00000 00
87	7	3.50000 00000 00
87	8	4.00000 00000 00
87	9	4.50000 00000 00
87	10	4.99999 99999 46
87	11	5.50000 13627 77
87	12	5.99009 65363 64
88	1	0.25000 00000 00
88	2	0.75000 00000 00
88	3	1.25000 00000 00
88	4	1.75000 00000 00
88	5	2.25000 00000 00
88	6	2.75000 00000 00
88	7	3.25000 00000 00
88	8	3.75000 00000 00
88	9	4.25000 00000 00
88	10	4.75000 00000 00
88	11	5.24999 99969 34
88	12	5.75004 89250 37
88	13	6.16952 42595 01
89	1	0.50000 00000 00
89	2	1.00000 00000 00
89	3	1.50000 00000 00
89	4	2.00000 00000 00
89	5	2.50000 00000 00
89	6	3.00000 00000 00
89	7	3.50000 00000 00
89	8	4.00000 00000 00
89	9	4.50000 00000 00
89	10	5.00000 00000 04
89	11	5.49999 98577 03
89	12	6.00155 50587 87
89	13	6.27112 45120 38
90	1	0.25000 00000 00
90	2	0.75000 00000 00
90	3	1.25000 00000 00
90	4	1.75000 00000 00
90	5	2.25000 00000 00
90	6	2.75000 00000 00
90	7	3.25000 00000 00
90	8	3.75000 00000 00
90	9	4.25000 00000 00
90	10	4.75000 00000 00
90	11	5.25000 00002 79
90	12	5.74999 44413 28
91	1	0.50000 00000 00
91	2	1.00000 00000 00
91	3	1.50000 00000 00
91	4	2.00000 00000 00
91	5	2.50000 00000 00
91	6	3.00000 00000 00
91	7	3.50000 00000 00
91	8	4.00000 00000 00
91	9	4.50000 00000 00
91	10	5.00000 00000 00
91	11	5.50000 00142 02
91	12	5.99981 41894 83
92	1	0.25000 00000 00
92	2	0.75000 00000 00
92	3	1.25000 00000 00
92	4	1.75000 00000 00
92	5	2.25000 00000 00
92	6	2.75000 00000 00
92	7	3.25000 00000 00
92	8	3.75000 00000 00
92	9	4.25000 00000 00
92	10	4.75000 00000 00
92	11	5.24999 99999 76
92	12	5.75000 06046 23
92	13	6.24499 67976 53
93	1	0.50000 00000 00
93	2	1.00000 00000 00
93	3	1.50000 00000 00
93	4	2.00000 00000 00
93	5	2.50000 00000 00
93	6	3.00000 00000 00
93	7	3.50000 00000 00
93	8	4.00000 00000 00
93	9	4.50000 00000 00
93	10	5.00000 00000 00
93	11	5.49999 99986 44
93	12	6.00002 19871 52
93	13	6.44386 24773 60

Table continued

TABLE V—*Continued*

n	r	Root
94	1	0.25000 00000 00
94	2	0.75000 00000 00
94	3	1.25000 00000 00
94	4	1.75000 00000 00
94	5	2.25000 00000 00
94	6	2.75000 00000 00
94	7	3.25000 00000 00
94	8	3.75000 00000 00
94	9	4.25000 00000 00
94	10	4.75000 00000 00
94	11	5.25000 00000 02
94	12	5.74999 99370 56
94	13	6.25070 25396 83
94	14	6.56141 10537 63
95	1	0.50000 00000 00
95	2	1.00000 00000 00
95	3	1.50000 00000 00
95	4	2.00000 00000 00
95	5	2.50000 00000 00
95	6	3.00000 00000 00
95	7	3.50000 00000 00
95	8	4.00000 00000 00
95	9	4.50000 00000 00
95	10	5.00000 00000 00
95	11	5.50000 00001 24
95	12	5.99999 75187 99
95	13	6.53585 37453 47
95	14	6.58383 06326 75
96	1	0.25000 00000 00
96	2	0.75000 00000 00
96	3	1.25000 00000 00
96	4	1.75000 00000 00
96	5	2.25000 00000 00
96	6	2.75000 00000 00
96	7	3.25000 00000 00
96	8	3.75000 00000 00
96	9	4.25000 00000 00
96	10	4.75000 00000 00
96	11	5.25000 00000 00
96	12	5.75000 00062 78
96	13	6.24991 55832 71
97		0.50000 00000 00
97	2	1.00000 00000 00
97	3	1.50000 00000 00
97	4	2.00000 00000 00
97	5	2.50000 00000 00
97	6	3.00000 00000 00
97	7	3.50000 00000 00
97	8	4.00000 00000 00
97	9	4.50000 00000 00
97	10	5.00000 00000 00
97	11	5.49999 99999 89
97	12	6.00000 02685 28
97	13	6.49757 71237 33
98	1	0.25000 00000 00
98	2	0.75000 00000 00
98	3	1.25000 00000 00
98	4	1.75000 00000 00
98	5	2.25000 00000 00
98	6	2.75000 00000 00
98	7	3.25000 00000 00
98	8	3.75000 00000 00
98	9	4.25000 00000 00
98	10	4.75000 00000 00
98	11	5.25000 00000 00
98	12	5.74999 99993 99
98	13	6.25000 98811 29
98	14	6.71348 36237 59
99	1	0.50000 00000 00
99	2	1.00000 00000 00
99	3	1.50000 00000 00
99	4	2.00000 00000 00
99	5	2.50000 00000 00
99	6	3.00000 00000 00
99	7	3.50000 00000 00
99	8	4.00000 00000 00
99	9	4.50000 00000 00
99	10	5.00000 00000 00
99	11	5.50000 00000 01
99	12	5.99999 99721 20
99	13	6.50031 93328 08
99	14	6.84808 04058 58
100	1	0.25000 00000 00
100	2	0.75000 00000 00
100	3	1.25000 00000 00
100	4	1.75000 00000 00
100	5	2.25000 00000 00
100	6	2.75000 00000 00
100	7	3.25000 00000 00
100	8	3.75000 00000 00
100	9	4.25000 00000 00
100	10	4.75000 00000 00
100	11	5.25000 00000 00
100	12	5.75000 00000 55
100	13	6.24999 88920 11
100	14	6.76090 71105 71
100	15	6.90862 45808 81
101	1	0.50000 00000 00
101	2	1.00000 00000 00
101	3	1.50000 00000 00
101	4	2.00000 00000 00
101	5	2.50000 00000 00
101	6	3.00000 00000 00
101	7	3.50000 00000 00
101	8	4.00000 00000 00
101	9	4.50000 00000 00
101	10	5.00000 00000 00
101	11	5.50000 00000 00
101	12	6.00000 00027 79
101	13	6.49996 17245 85
102	1	0.25000 00000 00
102	2	0.75000 00000 00
102	3	1.25000 00000 00
102	4	1.75000 00000 00
102	5	2.25000 00000 00
102	6	2.75000 00000 00
102	7	3.25000 00000 00
102	8	3.75000 00000 00
102	9	4.25000 00000 00
102	10	4.75000 00000 00
102	11	5.25000 00000 00
102	12	5.74999 99999 95
102	13	6.25000 01193 72
102	14	6.74885 57059 30

Table continued

TABLE V—*Continued*

n	r	Root
103	1	0.50000 00000 00
103	2	1.00000 00000 00
103	3	1.50000 00000 00
103	4	2.00000 00000 00
103	5	2.50000 00000 00
103	6	3.00000 00000 00
103	7	3.50000 00000 00
103	8	4.00000 00000 00
103	9	4.50000 00000 00
103	10	5.00000 00000 00
103	11	5.50000 00000 00
103	12	5.99999 99997 34
103	13	6.50000 44403 94
103	14	6.97807 18648 15
104	1	0.25000 00000 00
104	2	0.75000 00000 00
104	3	1.25000 00000 00
104	4	1.75000 00000 00
104	5	2.25000 00000 00
104	6	2.75000 00000 00
104	7	3.25000 00000 00
104	8	3.75000 00000 00
104	9	4.25000 00000 00
104	10	4.75000 00000 00
104	11	5.25000 00000 00
104	12	5.75000 00000 00
104	13	6.24999 99876 36
104	14	6.75014 54298 13
104	15	7.13127 72115 96
105	1	0.50000 00000 00
105	2	1.00000 00000 00
105	3	1.50000 00000 00
105	4	2.00000 00000 00
105	5	2.50000 00000 00
105	6	3.00000 00000 00
105	7	3.50000 00000 00
105	8	4.00000 00000 00
105	9	4.50000 00000 00
105	10	5.00000 00000 00
105	11	5.50000 00000 00
105	12	6.00000 00000 25
105	13	6.49999 95050 10
105	14	7.00456 02140 24
105	15	7.21145 02842 00
106	1	0.25000 00000 00
106	2	0.75000 00000 00
106	3	1.25000 00000 00
106	4	1.75000 00000 00
106	5	2.25000 00000 00
106	6	2.75000 00000 00
106	7	3.25000 00000 00
106	8	3.75000 00000 00
106	9	4..25000 00000 00
106	10	4.75000 00000 00
106	11	5.25000 00000 00
106	12	5.75000 00000 00
106	13	6.25000 00012 32
106	14	6.74998 26676 54
107	1	0.50000 00000 00
107	2	1.00000 00000 00
107	3	1.50000 00000 00
107	4	2.00000 00000 00
107	5	2.50000 00000 00
107	6	3.00000 00000 00
107	7	3.50000 00000 00
107	8	4.00000 00000 00
107	9	4.50000 00000 00
107	10	5.00000 00000 00
107	11	5.50000 00000 00
107	12	5.99999 99999 98
107	13	6.50000 00531 12
107	14	6.99946 70466 57
108	1	0.25000 00000 00
108	2	0.75000 00000 00
108	3	1.25000 00000 00
108	4	1.75000 00000 00
108	5	2.25000 00000 00
108	6	2.75000 00000 00
108	7	3.25000 00000 00
108	8	3.75000 00000 00
108	9	4.25000 00000 00
108	10	4.75000 00000 00
108	11	5.25000 00000 00
108	12	5.75000 00000 00
108	13	6.24999 99998 82
108	14	6.75000 19953 39
108	15	7.23784 71228 09
109	1	0.50000 00000 00
109	2	1.00000 00000 00
109	3	1.50000 00000 00
109	4	2.00000 00000 00
109	5	2.50000 00000 00
109	6	3.00000 00000 00
109	7	3.50000 00000 00
109	8	4.00000 00000 00
109	9	4.50000 00000 00
109	10	5.00000 00000 00
109	11	5.50000 00000 00
109	12	6.00000 00000 00
109	13	6.49999 99945 11
109	14	7.00006 62445 62
109	15	7.41084 86469 30
110	1	0.25000 00000 00
110	2	0.75000 00000 00
110	3	1.25000 00000 00
110	4	1.75000 00000 00
110	5	2.25000 00000 00
110	6	2.75000 00000 00
110	7	3.25000 00000 00
110	8	3.75000 00000 00
110	9	4.25000 00000 00
110	10	4.75000 00000 00
110	11	5.25000 00000 00
110	12	5.75000 00000 00
110	13	6.25000 00000 11
110	14	6.74999 97787 70
110	15	7.25202 93277 03
110	16	7.50724 89240 82
111	1	0.50000 00000 00
111	2	1.00000 00000 00
111	3	1.50000 00000 00
111	4	2.00000 00000 00
111	5	2.50000 00000 00
111	6	3.00000 00000 00
111	7	3.50000 00000 00
111	8	4.00000 00000 00
111	9	4.50000 00000 00
111	10	5.00000 00000 00
111	11	5.50000 00000 00
111	12	6.00000 00000 00
111	13	6.50000 00005 47
111	14	6.99999 21582 50

Table continued

TABLE V—*Continued*

n	r	Root	n	r	Root	n	r	Root
112	1	0.25000 00000 00	114	1	0.25000 00000 00	116	1	0.25000 00000 00
112	2	0.75000 00000 00	114	2	0.75000 00000 00	116	2	0.75000 00000 00
112	3	1.25000 00000 00	114	3	1.25000 00000 00	116	3	1.25000 00000 00
112	4	1.75000 00000 00	114	4	1.75000 00000 00	116	4	1.75000 00000 00
112	5	2.25000 00000 00	114	5	2.25000 00000 00	116	5	2.25000 00000 00
112	6	2.75000 00000 00	114	6	2.75000 00000 00	116	6	2.75000 00000 00
112	7	3.25000 00000 00	114	7	3.25000 00000 00	116	7	3.25000 00000 00
112	8	3.75000 00000 00	114	8	3.75000 00000 00	116	8	3.75000 00000 00
112	9	4.25000 00000 00	114	9	4.25000 00000 00	116	9	4.25000 00000 00
112	10	4.75000 00000 00	114	10	4.75000 00000 00	116	10	4.75000 00000 00
112	11	5.25000 00000 00	114	11	5.25000 00000 00	116	11	5.25000 00000 00
112	12	5.75000 00000 00	114	12	5.75000 00000 00	116	12	5.75000 00000 00
112	13	6.24999 99999 99	114	13	6.25000 00000 00	116	13	6.25000 00000 00
112	14	6.75000 00236 50	114	14	6.74999 99975 61	116	14	6.75000 00002 43
112	15	7.24975 36928 34	114	15	7.25003 01603 81	116	15	7.24999 64544 08
			114	16	7.68642 47996 82			
			115	1	0.50000 00000 00			
113	1	0.50000 00000 00	115	2	1.00000 00000 00	117	1	0.50000 00000 00
113	2	1.00000 00000 00	115	3	1.50000 00000 00	117	2	1.00000 00000 00
113	3	1.50000 00000 00	115	4	2.00000 00000 00	117	3	1.50000 00000 00
113	4	2.00000 00000 00	115	5	2.50000 00000 00	117	4	2.00000 00000 00
113	5	2.50000 00000 00	115	6	3.00000 00000 00	117	5	2.50000 00000 00
113	6	3.00000 00000 00	115	7	3.50000 00000 00	117	6	3.00000 00000 00
113	7	3.50000 00000 00	115	8	4.00000 00000 00	117	7	3.50000 00000 00
113	8	4.00000 00000 00	115	9	4.50000 00000 00	117	8	4.00000 00000 00
113	9	4.50000 00000 00	115	10	5.00000 00000 00	117	9	4.50000 00000 00
113	10	5.00000 00000 00	115	11	5.50000 00000 00	117	10	5.00000 00000 00
113	11	5.50000 00000 00	115	12	6.00000 00000 00	117	11	5.50000 00000 00
113	12	6.00000 00000 00	115	13	6.50000 00000 05	117	12	6.00000 00000 00
113	13	6.49999 99999 47	115	14	6.99999 99010 82	117	13	6.50000 00000 00
113	14	7.00000 08966 11	115	15	7.50092 29704 24	117	14	7.00000 00105 39
113	15	7.49370 57806 09	115	16	7.79852 56035 97	117	15	7.49988 66796 06

CO2AEF computations fell within these bounds. This was done using the original printout of the zeros of $B_n(x)$, $3 \leqslant n \leqslant 83$, to 32 decimal places.

A further check of all zeros is provided by the symmetry properties of both the real and complex zeros of $B_n(x)$. Replacing x by $(1+x)/2$ in (1.4) yields

$$(-1)^n B_n\left(-\frac{x}{2}-\frac{1}{2}\right) = B_n\left(\frac{x}{2}+\frac{1}{2}\right). \qquad (9.1)$$

Therefore, for *n even*, $B_n((1+x)/2)$ is an even function and, for *n odd*, $x^{-1}B_n((1+x)/2)$ is an even function. To obtain Table V we merely replace

x^2 by y (after factoring out x if n is odd) to obtain a polynomial of degree $[\![n/2]\!]$. This enabled us to obtain the zeros of $B_n(x)$, $84 \leqslant n \leqslant 117$, to 16 decimal places (we report 12 places in Table V). This second method of obtaining the zeros of $B_n(x)$ used an integration routine to obtain the coefficients of $B_n^*(x) = 2^n B_n((1+x)/2)$ since property (1.5) also holds for the polynomial set $\{B_n^*(x)\}$. This, along with the known symmetries of the real and complex zeros of $B_n(x)$, gave us another check on the accuracy of the zeros reported in Table IV.

We give here in Table IV a listing of the positive real zeros of $B_n(x)$, $3 \leqslant n \leqslant 83$, to 15 decimal places. In Table V we list the positive real zeros of $B_n(x)$, $84 \leqslant n \leqslant 117$, to 12 decimal places. A table of the complex zeros will appear elsewhere. Tables IV and V are a direct but reformatted printout of the zeros of $B_n(x)$ as generated by the modified CO2AEF routine. A copy of the original printouts is available from the author, upon request.

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